Artful Mathematics: The Heritage of M. C. Escher

Celebrating Mathematics Awareness Month

In recognition of the 2003 Mathematics Awareness Month theme "Mathematics and Art", this article brings together three different pieces about intersections between mathematics and the artwork of M. C. Escher. For more information about Mathematics Awareness Month, visit the website `http://mathforum.org/mam/03/`*. The site contains materials for organizing local celebrations of Mathematics Awareness Month.*

The Mathematical Structure of Escher's Print Gallery

B. de Smit and H. W. Lenstra Jr.

In 1956 the Dutch graphic artist Maurits Cornelis Escher (1898–1972) made an unusual lithograph with the title *Prentententoonstelling*. It shows a young man standing in an exhibition gallery, viewing a print of a Mediterranean seaport. As his eyes follow the quayside buildings shown on the print from left to right and then down, he discovers among them the very same gallery in which he is standing. A circular white patch in the middle of the lithograph contains Escher's monogram and signature.

What is the mathematics behind *Prentententoonstelling*? Is there a more satisfactory way of filling in the central white hole? We shall see that the lithograph can be viewed as drawn on a certain *elliptic curve* over the field of complex numbers and deduce that an idealized version of the picture repeats itself in the middle. More precisely, it contains a copy of itself, rotated clockwise by 157.6255960832... degrees and scaled down by a factor of 22.5836845286....

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Figure 1. Escher's lithograph "Prentententoonstelling" (1956).

Escher's Method

The best explanation of how *Prentententoonstelling* was made is found in *The Magic Mirror of M. C. Escher* by Bruno Ernst [1], from which the following quotations and all illustrations in this section are taken. Escher started "from the idea that it must...be possible to make an annular bulge," "a cyclic expansion...without beginning or end." The realization of this idea caused him "some almighty headaches." At first, he "tried to put his idea into

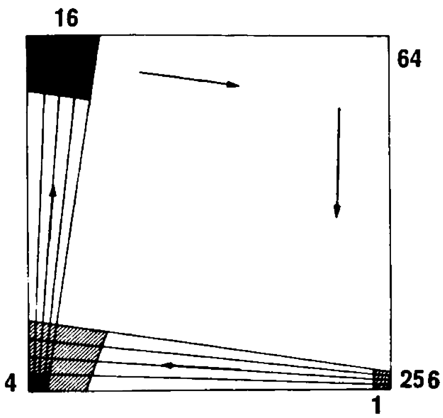

Figure 2. A cyclic expansion expressed using straight lines.

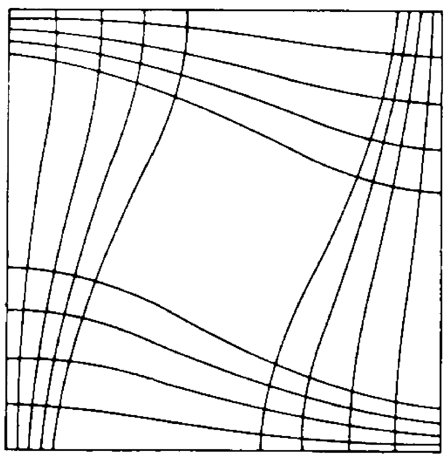

Figure 3. A cyclic expansion expressed using curved lines.

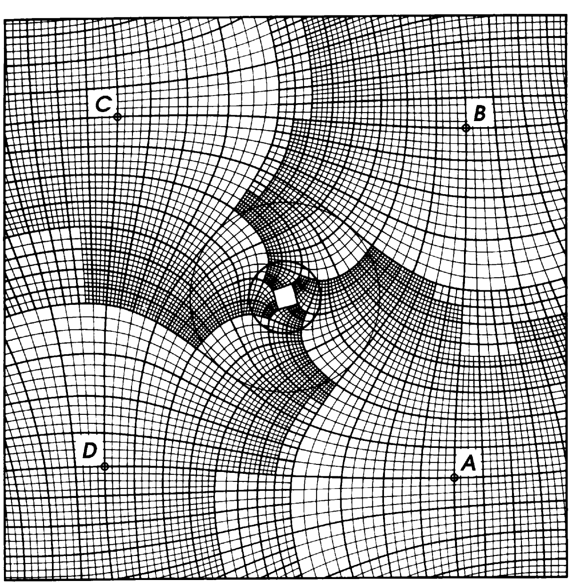

Figure 4. Escher's grid.

practice using straight lines [Figure 2], but then he intuitively adopted the curved lines shown in Figure [3]. In this way the original small squares could better retain their square appearance."

After a number of successive improvements Escher arrived at the grid shown in Figure 4. As one travels from A to D, the squares making up the grid expand by a factor of 4 in each direction. As one goes clockwise around the center, the grid folds onto itself, but expanded by a factor of $4^4 = 256$.

The second ingredient Escher needed was a normal, undistorted drawing depicting the same scene: a gallery in which a print exhibition is held, one of the prints showing a seaport with quayside buildings, and one of the buildings being the original

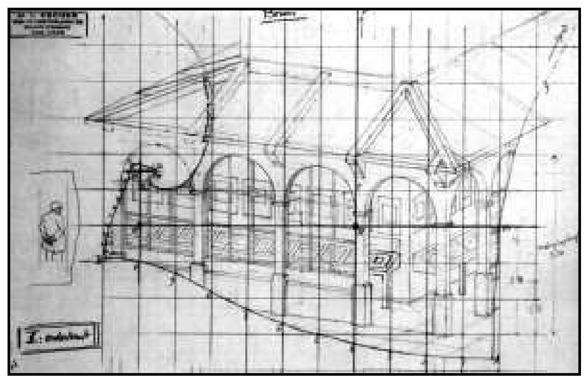

Figure 5. One of Escher's studies.

print gallery but reduced by a factor of 256. In order to do justice to the varying amount of detail that he needed, Escher actually made four studies instead of a single one (see [3]), one for each of the four corners of the lithograph. Figure 5 shows the study for the lower right corner. Each of these studies shows a portion of the previous one (modulo 4) but blown up by a factor of 4. Mathematically we may as well view Escher's four studies as a single drawing that is invariant under scaling by a factor of 256. Square by square, Escher then fitted the straight square grid of his four studies onto the curved grid, and in this way he obtained *Prententententoonstelling*. This is illustrated in Figure 6.

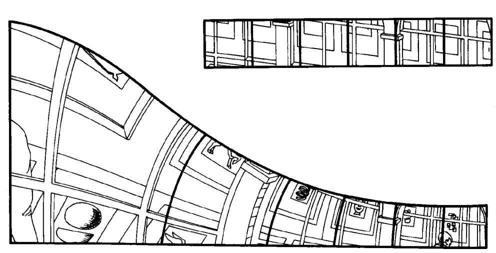

Figure 6. Fitting the straight squares onto the curved grid.

Below, we shall imagine the undistorted picture to be drawn on the complex plane $\mathbb{C}$, with 0 in the middle. We shall think of it as a function $f: \mathbb{C} \to \{\text{black, white}\}$ that assigns to each $z \in \mathbb{C}$ its color $f(z)$. The invariance condition then expresses itself as $f(256z) = f(z)$, for all $z \in \mathbb{C}$.

A Complex Multiplicative Period

Escher's procedure gives a very precise way of going back and forth between the straight world and the curved world. Let us make a number of walks on his curved grid and keep track of the corresponding walks in the straight world. First, consider the path that follows the grid lines from A to B to C to D and back to A. In the curved world this is a closed loop. The corresponding path, shown in Figure 7, in the straight world takes three left turns, each time travelling four times as far as the previous time before making the next turn. It is not a closed loop; rather, if the origin is properly chosen, the end point is 256 times the starting point. The same happens, with the same choice of origin, whenever one transforms a single closed loop, counterclockwise around the center, from the curved world to the straight world. It reflects the invariance of the straight picture under a blow-up by a factor of 256.

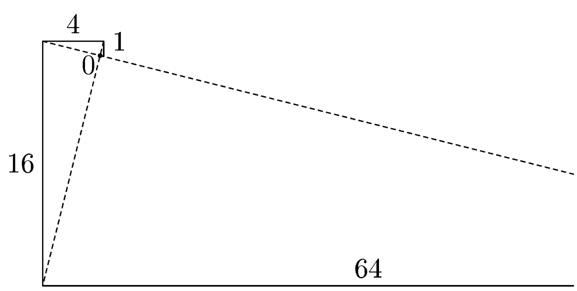

Figure 7. The square $ABCD$ transformed to the straight world.

No such phenomenon takes place if we do not walk around the center. For example, start again at A and travel 5 units, heading up; turn left and travel 5 units; and do this two more times. This gives rise to a closed loop in the curved world depicted in Figure 8, and in the straight world it corresponds to walking along the edges of a 5×5 square, another closed loop. But now do the same thing with 7 units instead of 5: in the straight world we again get a closed loop, along the edges of a 7×7 square, but in the curved world the path does not end up at A but at a vertex A' of the small square in the middle. This is illustrated in Figure 9. Since A and A' evidently correspond to the same point in the straight world, any picture made by Escher's procedure should, ideally, receive the same color at A and A'. We write *ideally*, since in Escher's actual lithograph A' ends up in the circular area in the middle.

New Escher Museum

In November 2002 a new museum devoted to Escher's works opened in The Hague, Netherlands. The museum, housed in the Palace Lange Voorhout, a royal palace built in 1764, contains a nearly complete collection of Escher's wood engravings, etchings, mezzotints, and lithographs. The initial exhibition includes major works such as *Day and Night*, *Ascending and Descending*, and *Belvedere*, as well as the *Metamorphoses* and self-portraits. The museum also includes a virtual reality tour that allows visitors to "ride through" the strange worlds created in Escher's works.

Further information may be found on the Web at `http://www.escherinhetpaleis.nl/`.

Allyn Jackson

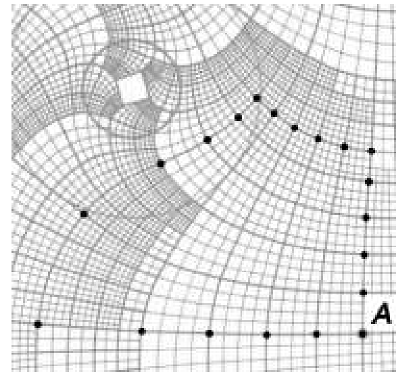

Figure 8. A 5 x 5 square transformed to the curved world.

We now identify the plane in which Escher's curved grid, or his lithograph, is drawn, also with $\mathbb{C}$, the origin being placed in the middle. Define $\gamma \in \mathbb{C}$ by $\gamma = A/A'$. A coarse measurement indicates that $|\gamma|$ is somewhat smaller than 20 and that the argument of γ is almost 3.

Replacing A in the procedure above by any point P lying on one of the grid lines AB, BC, CD, DA, we find a corresponding point P' lying on the boundary of the small square in the middle, and P' will ideally receive the same color as P. Within the limits of accuracy, it appears that the quotient

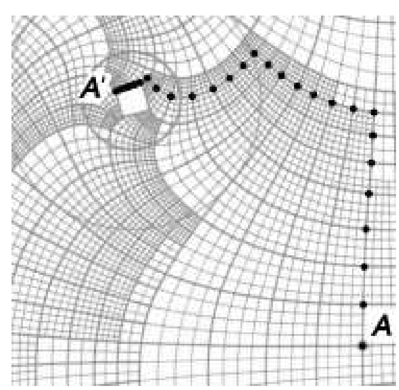

Figure 9. A 7 x 7 square transformed to the curved world.

P/P' is independent of P' and therefore also equal to γ. That is what we shall assume. Thus, when the "square" $ABCD$ is rotated clockwise over an angle of about 160° and shrunk by a factor of almost 20, it will coincide with the small central square.

Let the function g, defined on an appropriate subset of $\mathbb{C}$ and taking values in {black, white}, assign to each w its color $g(w)$ in Escher's lithograph. If Escher had used his entire grid—which, towards the middle, he did not—then, as we just argued, one would necessarily have $g(P') = g(\gamma P')$ for all P' as above, and therefore we would be able to extend his picture to all of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ by requiring $g(w) = g(\gamma w)$ for *all* w. This would not just fill in the hole *inside* Escher's lithograph but also the immense area that finds itself *outside* its boundaries.

Elliptic Curves

While the straight picture is periodic with a multiplicative period 256, an idealized version of the distorted picture is periodic with a complex period γ:

$$f(256z) = f(z), \qquad g(\gamma w) = g(w).$$

What is the connection between 256 and γ? Can one determine γ other than by measuring it in Escher's grid?

We start by reformulating what we know. For convenience, we remove 0 from $\mathbb{C}$ and consider functions on $\mathbb{C}^*$ rather than on $\mathbb{C}$. This leaves a hole that, unlike Escher's, is too small to notice. Next, instead of considering the function f with period 256, we may as well consider the induced function $\bar{f}: \mathbb{C}^*/\langle 256\rangle \to$ {black, white}, where $\langle 256\rangle$ denotes the subgroup of $\mathbb{C}^*$ generated by 256. Likewise, instead of g we shall consider $\bar{g}: \mathbb{C}^*/\langle\gamma\rangle \to$ {black, white}. Escher's grid provides the dictionary for going back and forth between f and g. A moment's reflection shows that all it does is provide a bijection $h: \mathbb{C}^*/\langle\gamma\rangle \xrightarrow{\sim} \mathbb{C}^*/\langle 256\rangle$ such that g is deduced from f by means of composition: $\bar{g} = \bar{f} \circ h$.

The key property of the map h is elucidated by the quotation from Bruno Ernst, "...the original squares could better retain their square appearance": Escher wished the map h to be a *conformal isomorphism*, in other words, an isomorphism of one-dimensional complex analytic varieties.

The structure of $\mathbb{C}^*/\langle\delta\rangle$, for any $\delta \in \mathbb{C}^*$ with $|\delta| \neq 1$, is easy to understand. The exponential map $\mathbb{C} \to \mathbb{C}^*$ induces a surjective conformal map $\mathbb{C} \to \mathbb{C}^*/\langle\delta\rangle$ that identifies $\mathbb{C}$ with the *universal covering space* of $\mathbb{C}^*/\langle\delta\rangle$ and whose kernel $L_\delta = \mathbb{Z}2\pi i + \mathbb{Z}\log\delta$ may be identified with the *fundamental group* of $\mathbb{C}^*/\langle\delta\rangle$. Also, we recognize $\mathbb{C}^*/\langle\delta\rangle$ as a thinly disguised version of the *elliptic curve* $\mathbb{C}/L_\delta$.

With this information we investigate what the map $h: \mathbb{C}^*/\langle\gamma\rangle \xrightarrow{\sim} \mathbb{C}^*/\langle 256\rangle$ can be. Choosing coordinates properly, we may assume that $h(1) = 1$. By algebraic topology, h lifts to a unique conformal isomorphism $\mathbb{C} \to \mathbb{C}$ that maps 0 to 0 and induces an isomorphism $\mathbb{C}/L_\gamma \to \mathbb{C}/L_{256}$. A standard result on complex tori (see [2, Ch. VI, Theorem 4.1]) now implies that the map $\mathbb{C} \to \mathbb{C}$ is a multiplication by a certain scalar $\alpha \in \mathbb{C}$ that satisfies $\alpha L_\gamma = L_{256}$. Altogether we obtain an isomorphism between two short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_\gamma & \longrightarrow & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^*/\langle\gamma\rangle & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & L_{256} & \longrightarrow & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^*/\langle 256\rangle & \longrightarrow & 0. \end{array}$$

Figure 10 illustrates the right commutative square.

In order to compute α, we use that the multiplication-by-α map $L_\gamma \to L_{256}$ may be thought of as a map between fundamental groups; indeed, it is nothing but the isomorphism between the fundamental groups of $\mathbb{C}^*/\langle\gamma\rangle$ and $\mathbb{C}^*/\langle 256\rangle$ induced

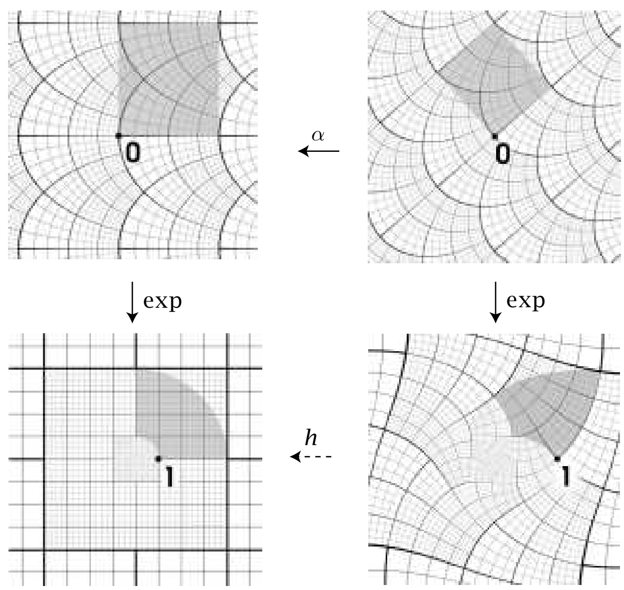

Figure 10. The picture on the lower left, drawn on $\mathbb{C}^*$, is invariant under multiplication by i and by 4. Pulled back to $\mathbb{C}$ by the exponential map it gives rise to a picture that is invariant under translation by the lattice $\frac{1}{4}L_{256} = \mathbb{Z}\pi i/2 + \mathbb{Z}\log 4$. Pulling this picture back by a scalar multiplication by $\alpha = (2\pi i + \log 256)/(2\pi i)$ changes the period lattice into $\frac{1}{4}L_\gamma = \mathbb{Z}\pi i/2 + \mathbb{Z}(\pi i \log 4)/(\pi i + 2\log 4)$. Since the latter lattice contains $2\pi i$, the picture can now be pushed forward by the exponential map. This produces the picture on the lower right, which is invariant under multiplication by all fourth roots of γ.
The bottom horizontal arrow represents the multivalued map $w \mapsto h(w) = w^\alpha = \exp(\alpha\log w)$; it is only modulo the scaling symmetry that it is well-defined.

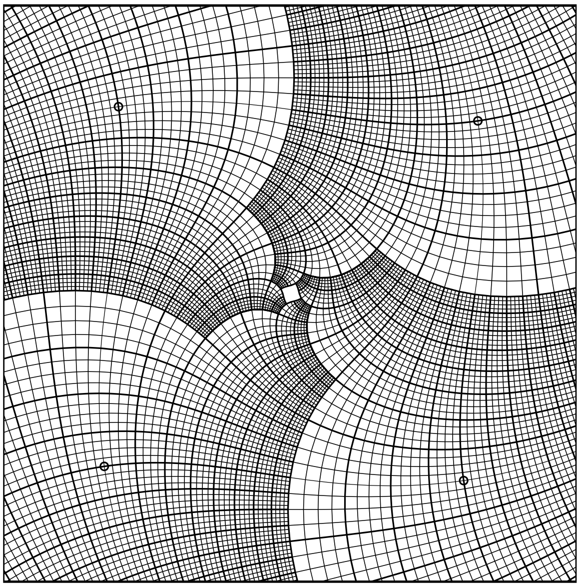

Figure 11. The perfectly conformal grid.

by h. The element $2\pi i$ in the fundamental group L_γ of $\mathbb{C}^*/\langle\gamma\rangle$ corresponds to a single counterclockwise loop around the origin in $\mathbb{C}^*$. Up to homotopy, it is the same as the path $ABCDA$ along grid lines that we considered earlier. As we saw, Escher's procedure transforms it into a path in $\mathbb{C}^*$ that goes once around the origin and at the same time multiplies by 256; in $\mathbb{C}^*/\langle 256\rangle$, this path becomes a closed loop that represents the element $2\pi i + \log 256$ of L_{256}. Thus, our isomorphism $L_\gamma \to L_{256}$ maps $2\pi i$ to $2\pi i + \log 256$, and therefore $\alpha = (2\pi i + \log 256)/(2\pi i)$. The lattice L_γ is now given by $L_\gamma = \alpha^{-1}L_{256}$, and from $|\gamma| > 1$ we deduce

$$\gamma = \exp(2\pi i(\log 256)/(2\pi i + \log 256))$$
$$\doteq \exp(3.1172277221 + 2.7510856371i).$$

The map h is given by the easy formula $h(w) = w^\alpha = w^{(2\pi i + \log 256)/(2\pi i)}$.

The grid obtained from our formula is given in Figure 11. It is strikingly similar to Escher's grid. Our small central square is smaller than Escher's; this reflects the fact that our value $|\gamma| \doteq 22.58$ is larger than the one measured in Escher's grid. The reader may notice some other differences, all of which indicate that Escher did not perfectly achieve his stated purpose of drawing a conformal grid, but it is remarkable how close he got by his own headache-causing process.

Filling in the Hole

In order to fill in the hole in *Prentenentoonstelling*, we first reconstructed Escher's studies from the grid and the lithograph by reversing his own procedure. For this purpose we used software specially written by Joost Batenburg, a mathematics student at Leiden. As one can see in Figure 12, the blank spot in the middle gave rise to an empty spiral in the reconstructed studies, and there were other imperfections as well. Next, the Dutch artists Hans Richter and Jacqueline Hofstra completed and adjusted the pictures obtained; see Figure 13.

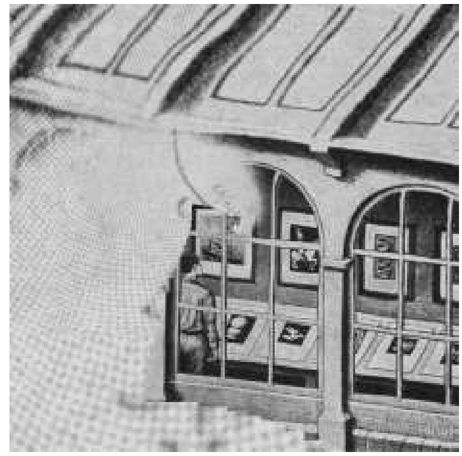

Figure 12. Escher's lithograph rectified, by means of his own grid.

When it came to adding the necessary grayscale, we ran into problems of discontinuous resolution and changing line widths. We decided that the natural way of overcoming these problems was by requiring the pixel density on our elliptic curve to be uniform in the Haar measure. In practical terms, we pulled the sketches back by the exponential function, obtaining a doubly periodic picture on $\mathbb{C}$; it was in that picture that the grayscale was added, by Jacqueline Hofstra, which resulted in the doubly periodic picture in Figure 14. The completed version of Escher's lithograph, shown in Figure 15, was then easy to produce.

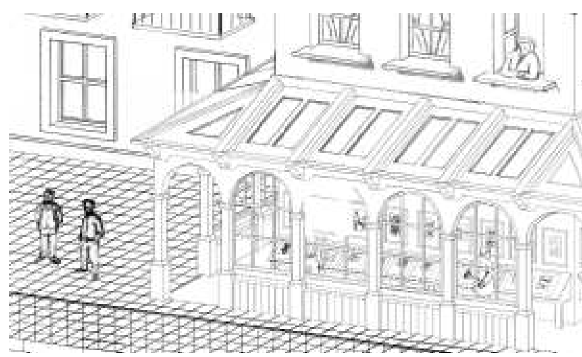

Figure 13. A detail of the drawing made by Hans Richter and Jacqueline Hofstra.

Other complex analytic maps $h: \mathbb{C}^*/\langle\delta\rangle \to \mathbb{C}^*/\langle 256\rangle$ for various δ give rise to interesting variants of *Prentenentoonstelling.* To see these

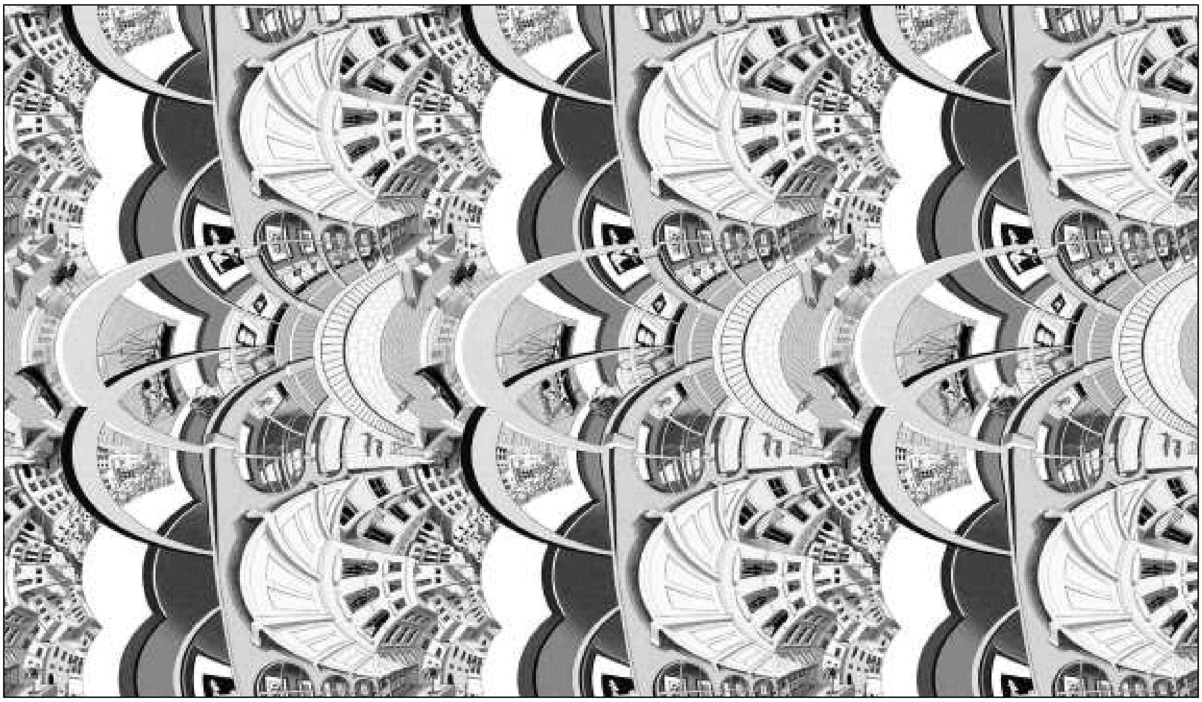

Figure 14. The straight drawing pulled back, by the complex exponential function, to a doubly periodic picture, with grayscale added. The horizontal period is log 256, the vertical period is $2\pi i$.

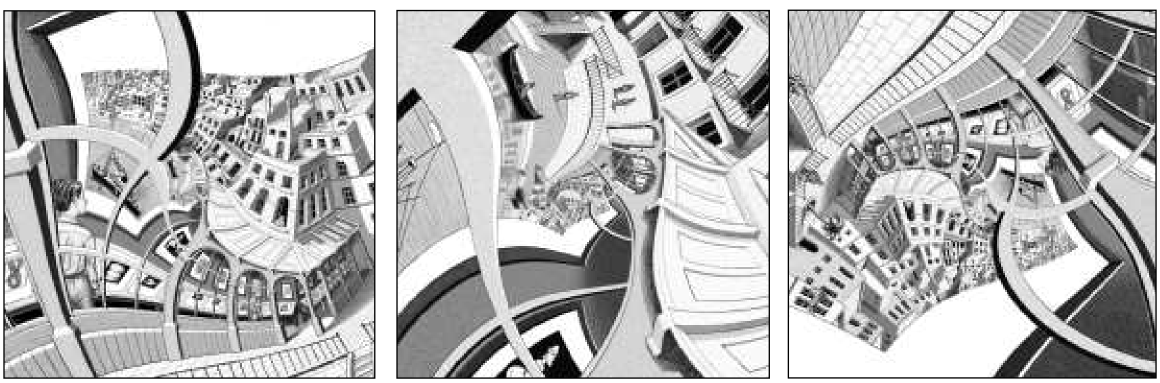

Figure 15. The completed version of Escher's lithograph with magnifications of the center by factors of 4 and 16.

and to view animations zooming in to the center of the pictures, the reader is encouraged to visit the website `escherdroste.math.leidenuniv.nl`.

Acknowledgments

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References

[1] Bruno Ernst, *De toverspiegel van M. C. Escher*, Meulenhoff, Amsterdam, 1976; English translation by John E. Brigham: *The Magic Mirror of M. C. Escher*, Ballantine Books, New York, 1976.

[2] J. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, New York, 1986.

[3] E. Thé (design), *The Magic of M. C. Escher*, Harry N. Abrams, New York and London, 2000.